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Analysis of inverse heat transfer for two-phase solidification problem in a finned phase-change material storage

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Abstract. In this paper, solution of the inverse heat conduction problems (IHCP) are presented in a finned phase-change material (PCM) storage and an imposed boundary condition type three on the vertical walls. At first direct heat transfer problem during solidification process were studied then based on sensitivity analysis the inverse problem solved numerically by Levenberg–Marquardt (LM) methods using temperature distribution and speed of freezing front defined by heat capacity methods, at least the temperature distribution on the boundary of PCM storage have been predicted.

Key words. Keywords: PCM, IHCP, sensitivity coefficients, Levenberg-Marquardt method.

1. Introduction

Uses of PCMs have expanded notably in two recent decades. Because of the low thermal conductivity of the PCMs, heat transfer enhancement techniques such as fins have to be used to increase the heat-transfer fraction in the store. Generally heat conductivity problems can be divided into the two direct and inverse categories. In the direct problems which are more applicable, geometry, thermo-physical properties and initial and boundary conditions are determined. In the inverse heat conductivity problems (IHCP) some of these data are unknown and extra information which are usually measured temperatures inside the solution area or on the boundary are known instead. In the solidification process, direct problem consists in a calculation of temperature distribution and freezing front position by simplified analytical models and numerical approaches, whilst inverse problem consists in a calculation of boundary conditions or other parameters using temperature distribution at a specified number of locations inside the domain. Several methods have

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been presented for solving an inverse heat transfer problem such that the problem is solved by minimizing a target function along an iterative procedure and in this article Levenberg–Marquardt [1, 2] method have been used. This technique is quite efficient for solving linear and nonlinear parameter estimation problems. However, difficulties may arise in nonlinear estimation problems involving a large number of unknown parameters, because of the time spent in the computation of the sensitivity matrix.

2. Direct problem

In the present work, the solidification process is investigated in a finite two-dimensional PCM storage with horizontal internal plate fins imposed boundary condition type three.

2.1. Analytical model

The heat equation for the PCM and enclosure with the initial and boundary conditions ere described by the equations

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_{\mathrm{p}}}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right],\ 0 \le x \le l_{\mathrm{f}},\ 0 \le y \le l_c,\ t \ge 0\,, \tag{1}$$

$$T(x, y, 0) = T_{\mathrm{i}}\,, \tag{2}$$

$$T(0, y, t) = T(l_{\mathrm{f}}, y, t) = T_{\mathrm{w}}\,. \tag{3}$$

Here, T_{i} is the initial temperature, T_{w} is the temperature of the wall, k denotes the thermal conductivity, ρ is the specific mass, C_{p} stands for the specific heat at a constant pressure. The energy balance for the solid–liquid interface with the initial conditions reads

$$\left[k_{\mathrm{s}}\left(\frac{\partial T}{\partial y}\right)_{\mathrm{s}} - k_{\mathrm{l}}\left(\frac{\partial T}{\partial y}\right)\right]\left[1 + \left(\frac{\partial S}{\partial x}\right)^2\right] = \rho_L \frac{\partial S}{\partial t}\,|_{y=S}\ , \tag{4}$$

where k_{s} is the thermal conductivity of the solid phase and k_{l} is the thermal conductivity of the liquid phase

2.2. Numerical model

The most commonly used numerical methods are the effective heat-capacity method and the enthalpy method. In this paper, the direct problem is solved numerically in two dimensions using the effective heat capacity method with a narrow temperature range, $\Delta T = 2\,^{\circ}\mathrm{C}$ [3, 4].

3. Definition of the inverse heat transfer problem

The target of an inverse problem is finding the inverse operatorQ^{-1} for which

$$Q\left[P(t)\right] = T(x,t) \rightarrow P(t) = Q^{-1}\left[T(x,t)\right] \tag{5}$$

such that, in the mentioned inverse problem, $P(t)$ is the unknown boundary temperature distribution.

3.1. Levenberg–Marquardt method for solving an inverse problem

Inverse problems for parameter estimation are usually solved by minimizing a target function along an iterative procedure. In this method, the target function S is defined as

$$S(P) = \sum_{i=1}^{l} \left[Y_i - T_i(P)\right]^2 , \tag{6}$$

where P is the vector of unknown parameters, T_i denotes the estimated temperature at time t_i and Y_i are the measured temperatures.

Levenberg–Marquardt method is quite efficient for solving linear and nonlinear parameter estimation problems. However, difficulties may arise in nonlinear estimation problems involving a large number of unknown parameters, because of the time spent in the computation of the sensitivity matrix. The solution of inverse parameter estimation by this technique requires the computation of the sensitivity matrix J, whose elements are the sensitivity coefficients J_{ij} defined as

$$J_{ij} = \frac{\partial T_i}{\partial P_j} , \tag{7}$$

where P_j is the jth unknown parameter (their number being N). The iterative procedure for finding vector P whose components are the unknown parameters of heat transfer problem will be as below:

$$P^{k+1} = P^k + \left[\left(J^k\right)^T J^k\right]^{-1} \left(J^k\right)^T \left[Y - T(P^k)\right] . \tag{8}$$

Problems satisfying $\left|JJ^T\right| = 0$ are very ill-conditioned. Inverse heat transfer problems are generally ill-conditioned, especially near the initial estimate used for the unknown parameters. The Levenberg–Marquardt method alleviates such difficulties by utilizing an iterative procedure in the form [5, 6]:

$$P^{k+1} = P^k + \left[\left(J^k\right)^{\mathrm{T}} J^k + \mu^k \Omega^k\right]^{-1} \left(J^k\right)^{\mathrm{T}} \left[Y - T\left(P^k\right)\right] , \tag{9}$$

where μ^k is a positive scalar called damping parameter and Ω^k is a diagonal matrix defined as

$$\Omega^k = \mathrm{diag}\left[\left(J^k\right)^{\mathrm{T}} J^k\right] . \tag{10}$$

The purpose of the matrix term $\mu^k\Omega^k$ included in the above equation is to damp oscillation and instabilities due to ill-conditioned character of the problem.

Suppose that temperature measurements $Y = (Y_1, Y_2, ..., Y_l)$ are given at times $t_i,\ i = 1, 2, ..., I$. Also an initial estimate P^0 is available for the vector of unknown parameters P. Choose a value for μ^0, say $\mu^0 = 0.001$. Now the algorithm of computation consists of the following steps:

Step 1. Solve the direct heat transfer problem given by equations (1) with the available estimate P^k in order to obtain the temperature vector $T\left(P^k\right) = (T_1, T_2, ..., T_L)$.

Step 2. Compute $S(P^k)$ from equation (6).

Step 3. Compute the sensitivity matrix J^k defined by equation (7) and then the matrix Ω^k given by equation (10), using the current values of P^k.

Step 4. Solve the following linear system of algebraic equations, obtained from the iterative procedure of the Levenberg–Marquardt method in order to compute the new estimate P^{k+1}:

$$P^{k+1} = P^k + \left[\left(J^k\right)^{\mathrm{T}} J^k + \mu^k\Omega^k\right]^{-1} \left(J^k\right)^{\mathrm{T}} \left[Y - T\left(P^k\right)\right] . \tag{11}$$

Step 5. Solve the direct problem with the new estimate P^{k+1} in order to find $T(P^{k+1})$. Then compute $S(P^{k+1})$, as defined by equation (11).

Step 6. If $S(P^{k+1}) \geq S(P^k)$, replace μ^k by $10\mu^k$ and return to Step 4.

Step 7. If $S(P^{k+1}) < S(P^k)$, accept the new estimate P^{k+1} and replace μ^k by $0.1\mu^k$.

Step 8. Check the stopping criteria given by the following equations (12 a, b, c). Stop the iterative procedure if any of them is satisfied. Otherwise, replace k by $k+1$ and return to Step 3.

$$S\left(P^{k+1}\right) \leq \varepsilon_1, \quad \left\|\left(J^k\right)^T \left[Y - T\left(P^k\right)\right]\right\| \leq \varepsilon_2, \quad \left\|P^{k+1} - P^k\right\| \leq \varepsilon_3\,, \tag{12}$$

where ε_1, ε_2 and ε_3 are selected tolerances.

4. Discussion and results

A scheme of the two dimensional symmetry cell of the PCM storage with an internal plate fin is shown in Fig. 1. The thermophysical properties of paraffin as a phase change material and aluminum as a fin metal are given in Table 1.

In this paper l_{f}, l_{c}, and fins thickness are 0.03 m, 0.01 m and 0.5 m, respectively. Furthermore, the initial temperature is 298.15 K, the material being considered in the liquid form. The boundary condition is of the convection type, the convection coefficient is assumed 10 W.m^{-2} K] and ambient temperature is 288.15 K. The optimized mesh has been found out 24×67. The resultant temperature distribution from solution of direct convection problem is shown in Figs. 2–5.

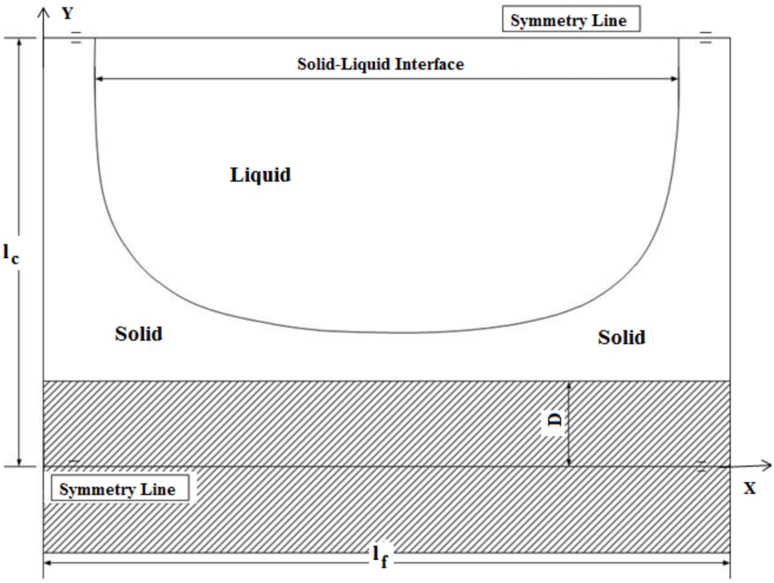

Fig. 1. Scheme of the two dimensional symmetry cell of the PCM storage

Table 1. Thermophysical properties of materials

Property	Paraffin	Aluminum
Density ($\mathrm{kg\,m^{-3}}$)	770	2710
Thermal conductivity ($\mathrm{W.m^{-1}K^{-1}}$)	0.185	174
Heat capacity, liquid ($\mathrm{J.kg^{-1}\,K^{-1}}$)	2400	-
Heat capacity, solid ($\mathrm{J.kg^{-1}\,K^{-1}}$)	1800	935
Latent heat of fusion ($\mathrm{J.kg^{-1}}$)	124098	-
Peak solidification temperature ($^\circ$C)	25	-

Fig. 2. Temperature distribution ($t = 200\,\mathrm{s}$), $T_{\min} = 288.15\,\mathrm{K}$, $T_{\max} = 298.15\,\mathrm{K}$, scale: light color$\longrightarrow$dark color means $T_{\min} \longrightarrow T_{\max}$

Our target is to find the temperature distribution along the boundaries. To achieve the results, we assume that the boundary condition is a third rank polynomial, si that we have 4 unknown parameters. By definition in equation (7) the

Fig. 3. Temperature distribution ($t = 400\,\mathrm{s}$), $T_{\min} = 288.15\,\mathrm{K}$, $T_{\max} = 292.21\,\mathrm{K}$, scale: light color⟶dark color means $T_{\min} \longrightarrow T_{\max}$

Fig. 4. Temperature distribution ($t = 600\,\mathrm{s}$), $T_{\min} = 288.15\,\mathrm{K}$, $T_{\max} = 289.61\,\mathrm{K}$, scale: light color⟶dark color means $T_{\min} \longrightarrow T_{\max}$

Fig. 5. Temperature distribution ($t = 600\,\mathrm{s}$), $T_{\min} = 288.15\,\mathrm{K}$, $T_{\max} = 288.32\,\mathrm{K}$, scale: light color⟶dark color means $T_{\min} \longrightarrow T_{\max}$

sensitivity factors are depicted in Figs. 6–9.

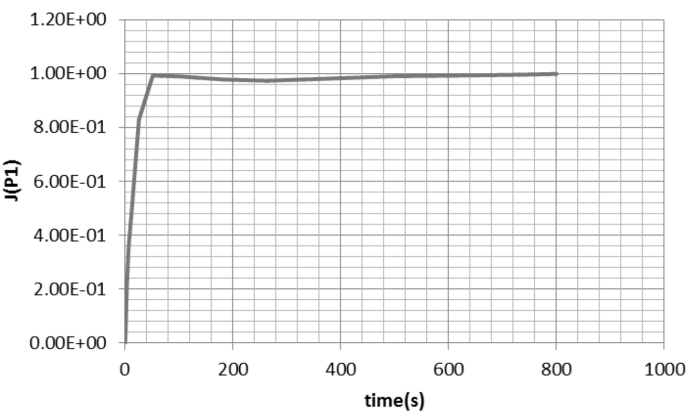

Fig. 6. Sensitivity factors for unknown parameter P_1

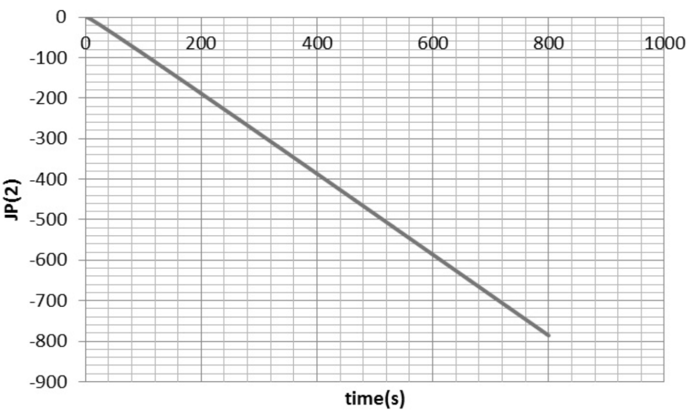

Fig. 7. Sensitivity factors for unknown parameter P_2

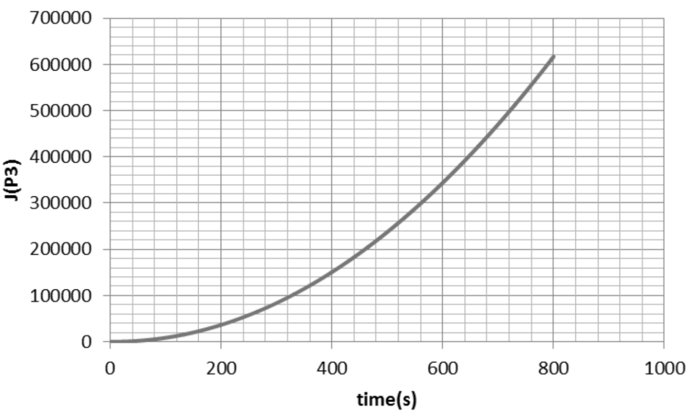

Fig. 8. Sensitivity factors for unknown parameter P_3

As illustrated, the resultant sensitivity factors shows no linear dependence so we could use Levenberg–Marquart freely. Furthermore, we could see that the resulting sensitivity factors have a growth in their scale, since there is a great difference between the unknown parameters. It could be expected to have large scales like in plots plot $J(p4)$ and $J(p3)$ because, as shown in equation (7), a small effect on unknown parameters could cause tremendous effect on sensitivity factor. Table 2 contains the initial estimates for unknown parameters.

Table 2. Initial estimates for unknown parameters

Unknown parameters	P_1	P_2	P_3	P_4
Initial guess	300	0.01	0.001	0.0001

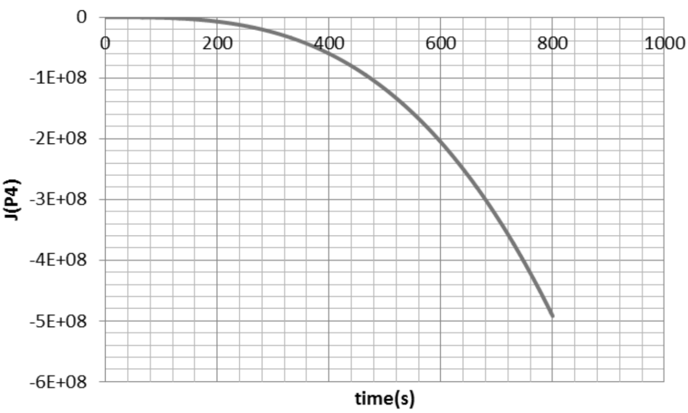

Fig. 9. Sensitivity factors for unknown parameter P_4

The information about solving sequence is presented in Table 3. Using the Levenberg–Marquart method we achieve equation (13):

$$T = -5 \cdot 10^{-9} t^3 + 9 \cdot 10^{-6} t^2 + 6.61 \cdot 10^{-3} t + 298.123 . \tag{13}$$

Table 3. Information about solving sequence

Error	Iteration number	Calculation time	Boundary condition
Initial guess	7	3 hours 37 minutes 12 seconds	convection

5. Conclusion

We show that by implementing the Levenberg–Marquardt method, in finned PCM storage case, we could solve inverse heat transfer problem to achieve unknown parameters of estimated polynomial function of boundary and obtained satisfactory results. We solve direct problem by heat capacity method, and used sensitivity analyses as a fundamental part of Levenberg–Marquardt method. Another strong method which we can use is Conjugate Lagrange method, which we suggest to calculate and compare with Levenberg–Marquardt method. This particular and strong method has no need to estimate function.

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